FRACTIONAL INTEGRATION OF PRODUCT OF GENERALIZED HYPERGEOMETRIC FUNCTION AND $\overline{\mathrm{H}}$ - FUNCTION

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ABSTRACT: *The object of this paper is to establish two theorems for generalized fractional integral operators applied on the product of Generalized Hypergeometric function and* $\overline{\mathrm{H}}$ *-function. The results are expressed in terms of* $\overline{\mathrm{H}}$ *- function.*

Keywords: Marichev Saigo Maeda Fractional Integral Operators, $\overline{\mathrm{H}}$ -Function, Generalized Hypergeometric Function

INTRODUCTION

Marichev-Saigo-Maeda Fractional Integral Operators

The generalized fractional integration operators of arbitrary order involving Appell function F3 in the kernel introduced by Marichev [1] and letter studied by Saigo and Maeda [2] in the following forms.

Let $\alpha,\alpha',\beta,\beta',\gamma,\rho \in$ and x>0 then,

$$\left(I_{0,+}^{\alpha,\alpha',\beta,\beta',\gamma} f\right)(x) = \frac{x^{-\mu}}{\Gamma(\gamma)}\int_0^x (x-t)^{\gamma-1} t^{-\alpha} \times F_3\left(\alpha,\alpha',\beta,\beta';1-\frac{t}{x},1-\frac{x}{t}\right) f(t)\,dt, \quad \text{…….(1)}$$

and $\mathrm{Re}(\gamma) > 0$

$$\left(I_{0,-}^{\alpha,\alpha',\beta,\beta',\gamma} f\right)(x) = \frac{x^{-\alpha'}}{\Gamma(\gamma)}\int_x^\infty (t-x)^{\gamma-1} t^{-\alpha} \times F_3\left(\alpha,\alpha',\beta,\beta';\gamma;1-\frac{x}{t},1-\frac{t}{x}\right) f(t)\,dt \quad \text{…. .(2)}$$

$\mathrm{Re}(\gamma) > 0$

The following two results given by Saigo are needed in the sequal;

(I) Let $\alpha,\alpha',\beta,\beta',\gamma,\rho \in$ be such that, $\mathrm{Re}(\gamma) > 0$ and $\mathrm{Re}(\rho) > \max[0, \mathrm{Re}(\alpha-\alpha'-\beta-\gamma)$, $\mathrm{Re}(\alpha'-\beta')]$, then;

$$\left(I_{0,+}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1}\right)(x) = \Gamma\left[\begin{matrix}\rho,\rho+\gamma-\alpha-\alpha'-\beta,\rho+\beta'-\alpha' \\ \rho+\beta',\rho+\gamma-\alpha-\alpha',\rho+\gamma-\alpha'-\beta\end{matrix}\right] x^{\rho-\alpha-\alpha'+\gamma-1} \quad \text{….. (3)}$$

where , $$\Gamma\begin{bmatrix} a,b,c \\ x,y,z \end{bmatrix} = \frac{\Gamma(a)\Gamma(b)\Gamma(c)}{\Gamma(x)\Gamma(y)\Gamma(z)}$$

(II) Let Let $\alpha,\alpha',\beta,\beta',\gamma,\rho \in \mathbb{C}$ be such that, $\mathrm{Re}(\gamma) > 0$ and

$\mathrm{Re}(\rho) < 1 + \min[\mathrm{Re}(-\beta), \mathrm{Re}(\alpha+\alpha'-\gamma), \mathrm{Re}(\alpha+\beta'-\gamma)]$, then;

$$\left(I_{0,-}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1}\right)(x) = \Gamma\begin{bmatrix} 1-\rho-\gamma+\alpha+\alpha', 1-\rho+\alpha+\beta'-\gamma, 1-\rho-\beta \\ 1-\rho, 1-\rho+\alpha+\alpha'+\beta-\gamma, 1-\rho+\alpha-\beta \end{bmatrix} x^{\rho-\alpha-\alpha'+\gamma-1} \qquad \ldots(4)$$

Hypergeometric Function:

The Gauss Hypergeometric function [6] is defined by;

$${}_2F_1(a,b;c;z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k \ k!} z^n, \qquad \ldots..(5)$$

where c is neither zero nor a negative integer, for convergence, $|z| < 1$.

Also if Re (c-a-b) > 0, Re (c) > Re (b) >0 ,then ;

$${}_2F_1(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)},$$

The Generalized Hypergeometric function [6] is defined by;

$${}_pF_q\begin{bmatrix} a_1, a_2, \ldots\ldots, a_p; \\ b_1, b_2, \ldots\ldots, b_q; \end{bmatrix} z = \sum_{k=0}^{\infty} \frac{(a_1)_k, \ldots\ldots, (a_p)_k}{(b_1)_k, \ldots\ldots, (b_q)_k} \cdot \frac{z^n}{k!} \qquad \ldots\ldots (6)$$

where no denominator parameter equal to zero or negative integer.

H-function

The H-function defined by C. Fox [3], in terms of Mellin-Barnes type of contour integral is defined as follows:

$$H_{p,q}^{m,n}[z] = H_{p,q}^{m,n}\left[z \middle| \begin{matrix} (a_1,\alpha_1), \ldots\ldots, (a_p,\alpha_p) \\ (b_1,\beta_1), \ldots\ldots, (b_q,\beta_q) \end{matrix}\right]$$

$$= \frac{1}{2\pi i}\int_L z^s \phi(s)\, ds \ , \qquad (z \neq 0) \qquad \ldots\ldots.(7)$$

where,

$$\phi(s)=\frac{\prod_{j=1}^{m}\Gamma\left(b_j-\beta_j s\right)\prod_{j=1}^{n}\Gamma\left(1-a_j+\alpha_j s\right)}{\prod_{j=m+1}^{q}\Gamma\left(1-b_j+\beta_j s\right)\prod_{j=n+1}^{p}\Gamma\left(a_j-\alpha_j s\right)}, \qquad \text{......(8)}$$

Here m,n,p,q are integers satisfying $0\le m\le q,\ 0\le n\le p;\ a_j\left(j=1,......,p\right)$ and $b_j\left(j=1,........q\right)$ are complex parameters, $\alpha_j\ge 0\left(j=1,........,p\right)$, $\beta_j\ge 0\left(j=1,......,q\right)$ are positive numbers. The contour integral (5) converges absolutely if,

$$T=\sum_{j=1}^{n}\alpha_j-\sum_{j=n+1}^{p}\alpha_j+\sum_{j=1}^{m}\beta_j-\sum_{j=m+1}^{q}\beta_j>0 \qquad \text{.... (9)}$$

and $\left|\arg z\right|<\frac{1}{2}\pi T$ ...(10)

$\overline{\mathrm{H}}$ -Function:

The $\overline{\mathrm{H}}$ -function was introduced by Inayat Hussain [4] and studied by Bushman and Shrivastava [5] is defined and represented in the following manner,

$$\overline{H}_{p,q}^{m,n}\left[z\right]=\overline{H}_{p,q}^{m,n}\left[z\middle|\begin{matrix}(a_j,\alpha_j;A_j)_{1,q},(a_j,\alpha_j)_{n+1,p}\\(b_j,\beta_j)_{1,m},(b_j,\beta_j;B_j)_{m+1,q}\end{matrix}\right]$$

$$=\frac{1}{2\pi i}\int_L z^s\overline{\phi}\left(s\right)ds \qquad \left(z\neq 0\right) \qquad \text{.(11)}$$

where,

$$\overline{\phi}(s)=\frac{\prod_{j=1}^{m}\Gamma\left(b_j-\beta_j s\right)\prod_{j=1}^{n}\left\{\Gamma\left(1-a_j+\alpha_j s\right)\right\}^{A_j}}{\prod_{j=m+1}^{q}\left\{\Gamma\left(1-b_j+\beta_j s\right)\right\}^{B_j}\prod_{j=n+1}^{p}\Gamma\left(a_j-\alpha_j s\right)} \qquad \text{.. (12)}$$

Here L is a contour starting at the point c-i∞ and terminating at the point c+i∞, $a_j\left(j=1,......,p\right)$ and $b_j\left(j=1,........q\right)$ are complex parameters, $\alpha_j\ge 0\left(j=1,........,p\right)$, $\beta_j\ge 0\left(j=1,......,q\right)$ and the exponents $A_j\left(j=1,........,n\right)$, $B_j\left(j=m+1,......,q\right)$ can take integer values.

Sufficient condition for absolute convergenceof the contour integral in (4) established by Buschman and Shrivastava [5]is given as follows ;

$$T=\sum_{j=1}^{m}\beta_j+\sum_{j=1}^{n}\left|A_j\alpha_j\right|-\sum_{j=m+1}^{q}\left|B_j\beta_j\right|-\sum_{j=n+1}^{p}\alpha_j>0 \quad .... (13)$$

and $$|\arg z|<\frac{1}{2}\pi T \quad ...(14)$$

Main Results:

Theorem 1: If $\alpha,\alpha',\beta,\beta',\rho\in\square$, $x>0,\ T>0,\ |\arg z|\ <\ \frac{1}{2}\pi T$, such that $\mathrm{Re}(\gamma)>0$ and $\mathrm{Re}[\rho+\eta k+\mu\xi]\ >\max[0,\ \mathrm{Re}(\alpha+\alpha'+\beta-\gamma),\ \mathrm{Re}(\alpha'-\beta')]$, then;

$$\mathbf{I}_{0,+}^{\alpha,\alpha',\beta,\beta',\gamma}\left(t^{\rho-1}\ {}_pF_q\left[\begin{matrix}a_1,a_2,.....,a_p;\\ b_1,b_2,.....,b_q;\end{matrix}\ \lambda t^{\eta}\right]\ \bar{H}_{P,Q}^{M,N}\left[\omega t^{\mu}\left|\begin{matrix}(e_j,E_j,A_j)_{1,N},(e_j,E_j)_{N+1,P}\\ (f_j,F_j)_{1,M},(f_j,F_j,B_j)_{M+1,q}\end{matrix}\right.\right]\right)(x)$$

$$= x^{\rho+\eta k-\alpha-\alpha'+\gamma-1}\sum_{k=0}^{\infty}\frac{(a_1)_k,........,(a_p)_k\ \lambda^k}{(b_1)_k,........,(b_q)_k\ k!}$$

$$\times\bar{H}_{P+3,Q+3}^{M,N+3}\left[\omega x^{\mu}\left|\begin{matrix}(1-\rho-\eta k,\mu;1),(1-\rho-\eta k-\gamma+\alpha+\alpha'+\beta,\mu;1),(1-\rho-\eta k-\beta'+\alpha',\mu;1),(e_j,E_j;A_j)_{1,N},(e_j,E_j)_{N+1,P}\\ (f_j,F_j)_{1,M},(f_j,F_j;B_j)_{M+1,Q},(1-\rho-\eta k-\beta',\mu;1),(1-\rho-\eta k-\gamma+\alpha+\alpha',\mu;1),(1-\rho-\eta k-\gamma+\alpha'+\beta,\mu;1)\end{matrix}\right.\right]$$

……...(15)

Proof : Applying equation (6) and (11) to the left hand side of (15) and then interchanging the order of summation and integration we have,

$$\mathbf{I}_{0,+}^{\alpha,\alpha',\beta,\beta',\gamma}\left(t^{\rho-1}\ {}_pF_q\left[\begin{matrix}a_1,a_2,.....,a_p;\\ b_1,b_2,.....,b_q;\end{matrix}\ \lambda t^{\eta}\right]\ \bar{H}_{P,Q}^{M,N}\left[\omega t^{\mu}\left|\begin{matrix}(e_j,E_j,A_j)_{1,N},(e_j,E_j)_{N+1,P}\\ (f_j,F_j)_{1,M},(f_j,F_j,B_j)_{M+1,Q}\end{matrix}\right.\right]\right)(x)$$

$$=\sum_{k=0}^{\infty}\frac{(a_1)_k,........,(a_p)_k\ \lambda^k}{(b_1)_k,........,(b_p)_k\ k!}\times\frac{1}{2\pi i}\int_L\omega^{\xi}\theta(\xi)\ \left\{I_{0,+}^{\alpha,\alpha',\beta,\beta',\gamma}\ t^{\rho+\eta k+\mu\xi-1}\right\}(x)\,d\xi$$

Now applying the Saigo Maeda operator (3) we obtain the right hand side of (15).

Theorem 2: If $\alpha, \alpha', \beta, \beta', \rho \in \square$, $x > 0,\ T > 0,\ |\arg z| < \frac{1}{2}\pi T$,such that $\mathrm{Re}(\gamma) > 0$ and $\mathrm{Re}[\rho + \eta k - \mu\xi] < 1 + \min[\ \mathrm{Re}(-\beta),\ \mathrm{Re}(\alpha + \alpha' - \gamma), (\alpha + \beta' - \gamma)]$, then;

$$\mathbf{I}_{0,-}^{\alpha,\alpha',\beta,\beta',\gamma}\left(t^{\rho-1}\ {}_pF_q\left[\begin{matrix} a_1, a_2, \ldots\ldots, a_p; \\ b_1, b_2, \ldots\ldots, b_q; \end{matrix}\ \lambda t^{\eta}\right]\ \bar{H}_{P,Q}^{M,N}\left[\omega\, t^{-\mu}\left| \begin{matrix} (e_j, E_j, A_j)_{1,N}, (e_j, E_j)_{N+1,P} \\ (f_j, F_j)_{1,M}, (f_j, F_j, B_j)_{M+1,q} \end{matrix} \right.\right]\right)(x)$$

$$x^{\rho+\eta k-\alpha-\alpha'+\gamma-1} \times \sum_{k=0}^{\infty} \frac{(a_1)_k, \ldots\ldots\ldots, (a_p)_k}{(b_1)_k, \ldots\ldots\ldots, (b_p)_k} \frac{\lambda^k}{k!}$$

$$\times \bar{H}_{P+3,Q+3}^{M,N+3}\left[\omega x^{-\mu}\left| \begin{matrix} (\rho+\eta k+\gamma-\alpha-\alpha', \mu; 1), (\rho+\eta k-\alpha-\beta'+\gamma, \mu; 1), (\rho+\eta k+\beta, \mu; 1), (e_j, E_j; A_j)_{1,N}, (e_j, E_j)_{N+1,P} \\ (f_j, F_j)_{1,M}, (f_j, F_j; B_j)_{M+1,Q}, (\rho+\eta k, \mu; 1), (\rho+\eta k-\alpha-\alpha'-\beta+\gamma', \mu; 1), (\rho+\eta k-\alpha+\beta, \mu; 1) \end{matrix} \right.\right]$$

……(16)

Proof : Applying equation (6) and (11) to the left hand side of (16) and then interchanging the order of summation and integration we have,

$$I_{-}^{\alpha,\alpha',\beta,\beta',\gamma}\left(t^{\rho-1}\ {}_pF_q\left[\begin{matrix} a_1, a_2, \ldots\ldots, a_p; \\ b_1, b_2, \ldots\ldots, b_q; \end{matrix}\ \lambda t^{\eta}\right]\ \bar{H}_{P,Q}^{M,N}\left[\omega t^{-\mu}\left| \begin{matrix} (e_j, E_j, A_j)_{1,N}, (e_j, E_j)_{N+1,P} \\ (f_j, F_j)_{1,M}, (f_j, F_j, B_j)_{M+1,q} \end{matrix} \right.\right]\right)(x)$$

$$= \sum_{k=0}^{\infty} \frac{(a_1)_k, \ldots\ldots\ldots, (a_p)_k}{(b_1)_k, \ldots\ldots\ldots, (b_q)_k} \frac{\lambda^k}{k!} \times \frac{1}{2\pi i} \int_L \omega^{\xi} \theta(\xi)\ \left\{ I_{0,+}^{\alpha,\alpha',\beta,\beta',\gamma}\ t^{\rho+\eta k-\mu\xi-1} \right\}(x)\, d\xi$$

Now applying the Saigo Maeda operator (4) we obtain the right hand side of (16).

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